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Spherically Symmetric Vacuum Brane and Wormhole Solutions

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Abstract In this letter, we have obtained static, spherically symmetric solutions of the effective vacuum Einstein field equations on a brane embedded in a five dimensional space time. The effective stress tensor is induced by the interaction with the bulk gravitational field and is given by the electric part of the five dimensional Weyl tensor. Due to traceless nature of this non-local effect of the bulk, any solution of ${}^{(4)}R = 0$ is a possible solution of the vacuum brane. We have derived a class of solutions, which corresponds to wormhole solution. Physical properties and characteristics of the wormhole are studied.

Keywords Braneworld · Wormhole · Spherical symmetry

The brane world scenario [1, 2] (for brief review see [3, 4]) is a four dimensional hypersurface embedded in a five dimensional bulk. From the view point of string theory [5, 6], the observed four dimensional world (the brane) is a kind of domain wall in a five dimensional space time. In the second Randall-Sundrum model [2], our universe (a three brane) is embedded in a five dimensional bulk, where the extra dimension can extend to infinity in either side of the brane. The standard model fields are confined to the brane while gravity can propagate in the surrounding bulk. The gravitational field on the brane is described by modified 4D Einstein equations derived by Shiromizu et al. [7] from the 5D bulk Einstein equations using Gauss and Codazzi equations. The modifications are done in the energy momentum tensor by introducing two correction terms. One correction term is quadratic in the energy momentum tensor on the brane and is termed as local bulk effect while the second correction term is the non-local effect of the bulk and is given by the electric part of the 5D Weyl tensor. Thus, when there is no matter in the brane, the effective Einstein equations simplify to (choosing $8\pi G = 1 = c$)

$$G_{\mu\nu} = -E_{\mu\nu} \tag{1}$$

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where the non-local bulk effect $E_{\mu\nu}$ is trace-free, i.e., $E^{\mu}_{\mu} = 0$. Due to the geometric origin, $E_{\mu\nu}$ does not necessarily satisfy the usual energy conditions obeyed by ordinary matter (for example, see [8]). So it is expected that $E_{\mu\nu}$ may be possible matter supporting wormholes on the brane.

Wormhole is a purely theoretical ingredient [9, 10], which is considered as a smooth bridge between different universes or between two remote parts of a single universe. The wormhole geometry is described as a solution of the Einstein equation, provided that the stress energy tensor of the matter violates the null energy condition at least in the vicinity of the wormhole throat. This theoretical idea gets lot of attention for the past few years due to the observational prediction [11, 12] that the universe is accelerating at present. Standard cosmology can explain this accelerating phase only by introducing some exotic or phantom-like [13] matter, which violates the standard energy conditions. So it is interesting to attempt for wormhole solution in the vacuum brane scenario where the matter term $E_{\mu\nu}$ has exotic behaviour.

In Schwarzschild coordinates, a spherically symmetric and static wormhole geometry is described by the space time metric

$$ds^2 = -\exp(2\Phi)\, dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega_2{}^2 \tag{2}$$

where the function $\Phi(r)$ is called the redshift function and $b(r)$ is known as the shape function. These two functions are not arbitrary but satisfy the following conditions to have a wormhole solution:

I. At the throat radius r_0, $b(r_0) = r_0$.

II. A flaring out condition of the throat i.e.,

$$(b - b'r)/b^2 > 0 \tag{3}$$

which at the throat simplifies to $b'(r_0) < 1$.

III. $b(r) < r$ for $r > r_0$.

IV. For traversable wormhole, there should not be any horizons present i.e., Φ must be finite everywhere.

Usually, the radial coordinate r ranges over $[r_0, \infty)$, but there may be a cut off of the stress energy tensor at any finite junction radius R, where the interior space time matches to an exterior vacuum solution.

If the exterior vacuum space time is chosen for simplicity as the Schwarzschild space time i.e.,

$$ds_E^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{(1 - \frac{2M}{r})} + r^2 d\Omega_2{}^2 \tag{4}$$

then one must have

$$R > r_0 = 2M \quad \text{(event horizon)}.$$

Using the junction conditions due to Darmois-Israel formalism [14, 15], the components of the surface stresses of a dynamic thin shell [16, 17] are

$$\sigma \text{ (surface energy density)} = -\frac{1}{4\pi R}\left[\sqrt{1 - \frac{2M}{R} + \dot{R}^2} - \sqrt{1 - \frac{b(R)}{R} + \dot{R}^2}\right] \tag{5}$$

and

$\mathcal{P}$ (tangential surface pressure)

$$= \frac{1}{8\pi R}\left[\frac{1-\frac{M}{R}+\dot{R}^2+R\ddot{R}}{\sqrt{1-\frac{2M}{R}+\dot{R}^2}} - \frac{(1+R\Phi')(1-\frac{b(R)}{R}+\dot{R}^2)+R\ddot{R}-\frac{\dot{R}^2(b-b'R)}{2(R-b)}}{\sqrt{1-\frac{b(R)}{R}+\dot{R}^2}}\right] \tag{6}$$

where the overdot denotes derivative with respect to the proper time τ.

Further, if $M_s = 4\pi r^2\sigma$ denotes the surface mass of the thin shell (for static case), then the total mass M_T of the wormhole has the expression [18]

$$M_T = \frac{b(R)}{2} + M_s\left[\sqrt{1-\frac{b(R)}{r}} - \frac{M_s}{2R}\right] \tag{7}$$

Also for the static case, if we have the boundary surface i.e., $\sigma = 0 = \mathcal{P}$, then one obtains the following relationships

$$b(R) = 2M \quad \text{and} \quad R = 2M\left(\frac{\zeta - 1/2}{\zeta - 1}\right) \tag{8}$$

As $R > 2M$, so the redshift parameter ζ $[= 1 + R\Phi'(R)]$ is restricted by $\zeta > 1$.

In the present work, we shall start with a general form of static spherically symmetric metric given by

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2 d\Omega_2{}^2 \tag{9}$$

As the brane has no matter, so the effective Einstein equations on the brane simplifies to (1), which on contraction gives ${}^{(4)}R = 0$ i.e.,

$$f(r)\frac{A'}{A} + \frac{2A}{r^2} + g(r) = 0 \tag{10}$$

where

$$f(r) = \frac{B'}{B} + \frac{2}{r} \quad \text{and} \quad g(r) = -\frac{B''}{B} + \frac{B'^2}{2B^2} - \frac{2}{r}\frac{B'}{B} - \frac{2}{r^2} \tag{11}$$

Now solution of (10) can be written as

$$A^{-1} = e^{\int \frac{g}{f}dr}\left[C + \int \frac{2}{fr^2}\, e^{-\int \frac{g}{f}dr}dr\right] \tag{12}$$

with C as an integration constant.

As we have only one equation (viz. (10)) containing two unknowns (A and B), so we must have to specify one of them. Here the simplest way is to choose B and then evaluate A from (12). In the following, some specific choices of B are given and the solutions are examined whether wormhole criteria is satisfied or not.

As the space time is static and spherically symmetric in nature, so the non-local bulk effects contribute in the form of an effective anisotropic fluid in the form [19]

$$E^{\nu}_{\mu} = \text{diag}[\epsilon(r), \sigma_r(r), \sigma_t(r), \sigma_t(r)]$$

with

$$-\epsilon + \sigma_r + 2\sigma_t = 0 \tag{13}$$

due to tracelessness of $E_{\mu\nu}$. Now for the metric (2), the $(t\text{–}t)$ and $(r\text{–}r)$ components of the Einstein field equations are

$$-\frac{\epsilon}{8\pi} = \frac{1}{r^2} - \frac{1}{A}\left(\frac{1}{r^2} - \frac{1}{r}\frac{A'}{A}\right) \tag{14}$$

$$-\frac{\sigma_r}{8\pi} = -\frac{1}{r^2} + \frac{1}{A}\left(\frac{1}{r^2} + \frac{1}{r}\frac{B'}{B}\right) \tag{15}$$

Thus we can estimate the components of $E_{\mu\nu}$ using (13)–(15).

Choice-I: $B = Kr^{\alpha}$ Then from (12) using (11), we have

$$A^{-1} = \frac{4}{n(\alpha+4)} + \frac{C}{r^n}, \quad n = \frac{\alpha^2 + 2\alpha + 4}{\alpha + 4},$$

$C =$ integration constant.

Now comparing with metric (2) we have

$$b(r) = r\left[1 - \frac{C}{r^n} - \frac{4}{n(\alpha+4)}\right]$$

Due to the conditions (3), the shape function becomes

$$b(r) = r - \frac{4r}{n(\alpha+4)}\left[1 - \left(\frac{r_0}{r}\right)^n\right] \tag{16}$$

with $\alpha > -4$.

Thus the solution describes a wormhole geometry, but as the solution is not asymptotically flat, so the non local bulk effect is confined in a finite region bounded by an exterior vacuum space time. Further, from (13)–(15) using (16), the components of $E_{\mu\nu}$ have explicit expressions

$$\epsilon = -\frac{8\pi}{r^2}\left[1 - \frac{4}{n(\alpha+4)}\left\{1 - (1-n)\left(\frac{r_0}{r}\right)^n\right\}\right] \tag{17}$$

$$\sigma_r = 8\pi\left[-\frac{4\alpha}{n(\alpha+4)r^2}\left\{1 - \left(\frac{r_0}{r}\right)^n\right\} + \frac{1}{r^2}\left\{1 - \frac{4}{n(\alpha+4)}\left[1 - \left(\frac{r_0}{r}\right)^n\right]\right\}\right] \tag{18}$$

$$\sigma_t = -\frac{8\pi}{r^2} + \frac{16\pi(\alpha+2)}{n(\alpha+4)r^2} - \frac{16\pi(\alpha+2-n)}{n(\alpha+4)r^2}\left(\frac{r_0}{r}\right)^n \tag{19}$$

Choice-II: $B = \frac{e^{Kr}}{r^4}$ For this choice, $g(r) = 0$ and $f(r) = K/2$ and solution for A has the expression

$$A^{-1} = C - \frac{4}{Kr}$$

As before, in order to fulfill the criteria (3), the form of the shape function is

$$b(r) = r\left[1 - \frac{4}{Kr_0}\left(1 - \frac{r_0}{r}\right)\right] \tag{20}$$

Again the space time is not asymptotically flat, so the wormhole geometry is confined in a finite region across the throat. The expressions for the components of the non local bulk effect are

$$\epsilon = \frac{8\pi}{r^2}\left[\frac{4}{r_0 K} - 1\right] \tag{21}$$

$$\sigma_r = 8\pi\left[\frac{4}{Kr^3} + \frac{1}{r^2}\left(1 - \frac{4}{r_0 K}\right) - \frac{4}{Kr}\left(\frac{1}{r_0} - \frac{1}{r}\right)\left(K - \frac{4}{r}\right)\right] \tag{22}$$

$$\sigma_t = \frac{8\pi}{r}\left[\frac{6}{Kr^2} + \frac{2}{r_0} - \frac{3}{r} - \frac{4}{Krr_0}\right] \tag{23}$$

Choice-III: $B = r^2 e^{\alpha r}$ This choice of the scale factor B determines the other scale factor in an integral form as

$$A^{-1} = \frac{e^{-\alpha r}}{r^2}\left[C + \frac{4e^{\alpha r}}{\alpha^2} - \frac{24}{\alpha^2 e^6}\int \frac{e^z}{z}\,dz\right]$$

where $z = \alpha r + 6$.

The shape function takes the form

$$b(r) = r - \frac{4}{\alpha^2 r}\left[1 - e^{\alpha(r_0 - r)}\right] + \frac{24e^{-\alpha r}}{\alpha^2 r e^6}\left[E(z) - E(z_0)\right] \tag{24}$$

where $E(z) = \int \frac{e^z}{z}\,dz$.

Now, in order to satisfy the traversability criteria, α should be restricted as

$$\alpha > -\frac{6}{r_0} \tag{25}$$

The expressions for the induced matter components are

$$\begin{aligned}\epsilon = \frac{8\pi}{r^2}\Bigg[&\frac{4}{\alpha r}\,e^{\alpha(r_0 - r)}\left(1 + \frac{1}{\alpha r}\right) - \left(1 + \frac{4}{\alpha^2 r^2}\right)\\ &- \frac{24e^{-\alpha r}}{\alpha r e^6}\left\{E'(z) + \left[E(z_0) - E(z)\right]\left(1 + \frac{1}{\alpha r}\right)\right\}\Bigg]\end{aligned} \tag{26}$$

$$\sigma_r = \frac{8\pi}{r^2}\left[1 + \frac{4}{\alpha^2}(1 + 2r + r^2\alpha)\left\{\frac{6e^{-\alpha r}}{e^6}\left[E(z) - E(z_0)\right] - \left(1 - e^{\alpha(r_0 - r)}\right)\right\}\right] \tag{27}$$

$$\begin{aligned}\sigma_t = \frac{4\pi}{r^2}\Bigg[&\frac{4}{\alpha r}\,e^{\alpha(r_0 - r)}\left(1 + \frac{1}{\alpha r}\right) - \left(1 + \frac{4}{\alpha^2 r^2}\right)\\ &- \frac{24e^{-\alpha r}}{\alpha r e^6}\left\{E'(z) + \left[E(z_0) - E(z)\right]\left(1 + \frac{1}{\alpha r}\right)\right\}\Bigg]\\ &- \frac{4\pi}{r^2}\left[1 + \frac{4}{\alpha^2}(1 + 2r + r^2\alpha)\left\{\frac{6e^{-\alpha r}}{e^6}\left[E(z) - E(z_0)\right] - \left(1 - e^{\alpha(r_0 - r)}\right)\right\}\right]\end{aligned} \tag{28}$$

Choice-IV: $g(r) = -\frac{1}{r} f(r)$ For this choice, both the metric coefficients have the expressions

$$B = (d_1 - d_0/\sqrt{r})^2 \tag{29}$$

and

$$A^{-1} = \frac{1}{r}\left[C + r - \frac{d_0\sqrt{r}}{2d_1} - \frac{3d_0^2}{8d_1^2}\log\left|3d_0 - 4\sqrt{r}d_1\right|\right] \tag{30}$$

This solution will be a wormhole solution provided that the two parameters are restricted as

$$\frac{d_0}{d_1} < \sqrt{r_0}$$

The explicit form of the shape function is

$$b(r) = r_0 + \frac{d_0\sqrt{r}}{2d_1}\left(1 - \sqrt{\frac{r_0}{r}}\right) + \frac{3d_0^2}{8d_1^2}\log\left|\frac{3d_0 - 4d_1\sqrt{r}}{3d_0 - 4d_1\sqrt{r_0}}\right| \tag{31}$$

As the solution is asymptotically flat, so the wormhole geometry extends over the range $[r_0, \infty)$. The components of the stress-energy tensor are given by

$$\epsilon = \frac{8\pi d_0}{r^2(3d_0 - 4d_1\sqrt{r})} \tag{32}$$

$$\begin{aligned}\sigma_r = {} & \frac{8\pi}{r^3}\left[r_0 + \frac{d_0\sqrt{r}}{2d_1}\left(1 - \sqrt{\frac{r_0}{r}}\right) + \frac{3d_0^2}{8d_1^2}\log\left|\frac{3d_0 - 4d_1\sqrt{r}}{3d_0 - 4d_1\sqrt{r_0}}\right|\right] - \frac{8\pi d_0}{(d_1\sqrt{r} - d_0)r^3} \\ & \times \left\{1 - \frac{r_0}{r} - \frac{d_0}{2d_1\sqrt{r}}\left(1 - \sqrt{\frac{r_0}{r}}\right) - \frac{3d_0^2}{8d_1^2 r}\log\left|\frac{3d_0 - 4d_1\sqrt{r}}{3d_0 - 4d_1\sqrt{r_0}}\right|\right\}\end{aligned} \tag{33}$$

$$\begin{aligned}\sigma_t = {} & \frac{4\pi d_0}{r^2(3d_0 - 4d_1\sqrt{r})} - \frac{4\pi}{r^3}\left[r_0 + \frac{d_0\sqrt{r}}{2d_1}\left(1 - \sqrt{\frac{r_0}{r}}\right)\right. \\ & \left. + \frac{3d_0^2}{8d_1^2}\log\left|\frac{3d_0 - 4d_1\sqrt{r}}{3d_0 - 4d_1\sqrt{r_0}}\right|\right] + \frac{4\pi d_0}{(d_1\sqrt{r} - d_0)r^3}\left\{1 - \frac{r_0}{r}\right. \\ & \left. - \frac{d_0}{2d_1\sqrt{r}}\left(1 - \sqrt{\frac{r_0}{r}}\right) - \frac{3d_0^2}{8d_1^2 r}\log\left|\frac{3d_0 - 4d_1\sqrt{r}}{3d_0 - 4d_1\sqrt{r_0}}\right|\right\}\end{aligned} \tag{34}$$

Thus we have presented vacuum spherically symmetric brane solutions for four different choices and all are found to describe wormhole geometry. The generalization of Schwarzschild's solution to the braneworld framework was first done by Chamblin, Hawking and Reall [20] in their famous black string solution. Subsequently, Vollick [21] initiated the study of vacuum spherically symmetric brane and obtained two solutions—a Reissner-Nordström solution in GR (first mentioned by Dadhich et al. [22]) and a modified Schwarzschild type solution. The second one indeed describes a wormhole solution as first mentioned by Bronnikov et al. [23]. It should be mentioned that this modified Schwarzschild type solution was obtained earlier by Germani and Maartens [24] in studying the vacuum exterior of a spherical star in brane scenario and also by Casadio et al. [25] in search for new black holes in brane world.

A large class of static, spherically symmetric Lorentzian wormhole solutions were presented by Bronnikov and Kim [23] as a solution of ${}^{(4)}R = 0$ and were classified according to different properties. In fact, they formulated explicit algorithms for obtaining valid wormhole solutions. In this work, we have presented four possible wormhole solutions which are distinct from the above. In first three, we have specific choices for the metric coefficient B while in the fourth one, implicitly, a differential equation in B is assumed. In the first three cases, the wormhole solutions are confined to a finite region bounded by a vacuum exterior as the solutions are not asymptotically flat. On the other hand, the fourth solution is extended over the entire range $[r_0, \infty)$ and is a possible wormhole solution. For future work, it will be interesting to examine wormhole solution in brane scenario with matter content.

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